Learning to compare ratios in second grade: A path to avoid the natural number bias?

Aprender a comparar razones numéricas en segundo grado: ¿Una vía para evitar el sesgo de los números naturales?

David M. Gómez y Pablo Dartnell

Universidad de Chile

Abstract

A robust understanding of rational numbers requires rich notions of ratio and proportionality. However, previous research showed that children exhibit biases towards reasoning exclusively based on natural numbers when asked to compare fractions, ignoring the involved ratios. It is unknown the extent to which these biases depend on teaching methods. The present work thus assessed the emergence of biased reasoning in a context that highly emphasizes reasoning about the ratios. Forty second-grade children were taught to use fraction-like symbols to represent ratios, and then evaluated on a ratio comparison task. The acquired intuitive knowledge about ratios, on average, allowed successful ratio comparison. A clustering analysis revealed the presence of three groups of children using distinct patterns of reasoning. These data uncover important individual differences among children in intuition-based mental calculation about ratios, with implications for the teaching of ratios and fractions.

Keywords: numbers, mathematics education, fractions, teaching and learning.

Resumen

Un aprendizaje robusto de los números racionales requiere nociones ricas sobre razón y proporcionalidad. No obstante, la literatura muestra sesgos por parte de niños y niñas hacia razonar exclusivamente basados en números naturales cuando deben comparar fracciones, ignorando las razones subyacentes. No es claro hasta qué punto estos sesgos dependen de los métodos de enseñanza. Este trabajo evaluó la emergencia de sesgos en un contexto que favorece concentrarse sobre las razones. Cuarenta niños y niñas de segundo grado aprendieron a usar símbolos tipo fracciones para representar razones, respondiendo luego una tarea de comparación de razones. El conocimiento intuitivo adquirido sobre razones fue, en promedio, exitoso para la tarea de comparación. Un análisis de agrupación reveló la presencia de tres grupos que usan estrategias fundamentalmente distintas de pensamiento. Estos datos muestran importantes variaciones individuales sobre intuiciones para la comparación de razones, con implicancias para la enseñanza de razones y fracciones.

Palabras clave: números racionales, educación matemática, fracciones, enseñanza y aprendizaje.

Rational numbers are a crucial concept in elementary and middle school mathematics and at the same time one of the most difficult achievements, requiring a conceptual shift regarding what numbers are. In line with this high difficulty, research has linked rational number understanding to previous and future mathematics achievement (e.g. Aksu, 1997; Booth & Newton, 2012; Siegler et al., 2012). Still, many educators may not be fully prepared to teach fractions: Depaepe and colleagues (2015) showed that prospective teachers' knowledge of fractions reached a meager average of 79% (ranging from 34% to 98%) when tested with a questionnaire appropriate for upper elementary school according to the curriculum (see also Izsák, Orrill, Cohen, & Brown, 2010; Van Steenbrugge, Lesage, Valcke, & Desoete, 2014). One of the first pedagogical questions to deal with for teaching rational numbers is how to approach them, with several possible interpretations or metaphors available to the educator. Rational numbers can be represented as partitions and parts of an object, as positions in the number line, or as ratio-based relations between two quantities, among others. Educators often emphasize one or some of these representations over the others, and as a result the view of fractions as parts of objects is very common, whereas ratio-based relations and a ratio-based approach to fractions tend to be introduced much later, or sometimes simply neglected. Despite ratio and proportions being topics widely recognized as important and part of most mathematics curricula worldwide (see Obando, Vasco, & Arboleda, 2014, and references therein), their late introduction to students misses the opportunity to take advantage of children's intuitive understanding about ratios (e.g. Singer, Kohn, & Resnick, 1997; Van Den Brink & Streefland, 1979).

Several researchers in Mathematics Education and Psychology have studied a phenomenon named Natural Number Bias (hereafter NNB), consisting on the overgeneralization of concepts and intuitions proper of natural numbers to rationals (e.g. Ni & Zhou, 2005; Van Dooren, Lehtinen, & Verschaffel, 2015). This bias is evident in students' responses to, for instance, fraction comparison items like 3/5 vs. 3/7 and 5/9 vs. 7/9. The latter example is systematically found to be easier than the former (e.g. Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Eeckhoudt, 2013), because the magnitude of the relevant components ($5 < 7$) points either in the opposite ($3/5 > 3/7$) or the same ($5/9 < 7/9$) direction as the fractions' magnitudes. In line with previous studies, this paper will consider as congruent those fraction pairs in which the largest fraction is the one with the largest denominator and/or denominator, and incongruent those pairs where the largest fraction is the one with the smallest numerator and/or denominator (e.g. Gómez et al., 2014; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi et al., 2012). Examples of congruent and incongruent fraction pairs are presented in Table 1. The NNB has been found in the vast majority of studies about fraction comparison and fraction knowledge more generally, and even traces of it may be seen in expert mathematicians' response times to compare fractions (Obersteiner et al., 2013). It is so far an open question the extent to which the emergence of the NNB depends on the pedagogical strategies or approach used to teach fractions (Ni & Zhou, 2005).

The present work had a dual goal: First, to assess second grade children's ability to use their intuitive knowledge about ratios to respond to a ratio comparison task akin to the fraction comparison tasks documented in the literature. Second, by constructing and presenting both congruent and incongruent ratio pairs, this work evaluated the emergence of a NNB in this novel setting. To do so, a series of brief audiovisual recordings were developed and presented to second grade children to teach them the use of fraction-like symbols. These symbols (Figure 1) represented messages stating the number of candies per day that the inhabitants of a fictional land receive. This way, children were asked to compare ratios by judging which one of two of these messages was "more convenient" (this way of presenting fractions has been already studied by Jiménez, 2012, with fourth grade children). Such question falls into the associated sets format studied by Lamon (1993), consisting in creating ratios by pairing objects from different sets at a fixed rate (another examples are the ratios that naturally arise between sets of people and pizzas, or aliens and spaceships). Lamon interviewed sixth-grade children who had not received formal instruction about ratios so far, presenting them with problems about ratio and proportion in different formats. She found that problems presented in the associated sets format were most frequently solved by using some sort of qualitative proportional reasoning.

This work thus aims at exploring the feasibility and utility of using children's intuitions about ratios in order to avoid the NNB and achieve successful comparison of ratios, as a possible way to improve the teaching of fractions and rational numbers in general.

Figura 1. Example of a message used to represent ratios in the present study. In the context of the story, receiving this message means that its recipient would be given two candies every five days.

Method

Participants

Forty Chilean children (19 boys and 21 girls), approximately 7-8 years old, participated in this study. They were recruited from the second grade classes of a middle class school in the periphery of the capital city. Signed informed consent for participation was obtained from a parent of each participant prior to the testing session.

Material

Audiovisual recordings. Five audiovisual recordings were designed and developed. These videos presented the story of a fictional land, where elves bring messages and candies to children during the night. The reception of a message like the one in Figure 1 meant that the recipient would get two candies every five days. Two families of elves (yellow and green) were introduced, in order to give children the possibility of choosing which one of two messages was more convenient for them. Average length of the recordings was 3:12.

Ratio comparison task. Children also answered twelve ratio comparison questions, presented on the computer screen. In addition to having both congruent and incongruent items, the task included also items in which the two ratios shared the same number of days or the same number of candies. Table 1 shows the full list of items of this task.

Procedure

Children were tested in the Computer Science classroom of their school, in groups of between 20 to 30 children. Each child worked individually with a computer in a single testing session. One researcher and two teachers from the school were present during the session. After an introductory explanation of the content of the session, children watched the five audiovisual recor-

Tabla 1. List of items of the ratio comparison task, classified by item type.

Same number of days (congruent)	Same number of candies (incongruent)	All numbers different (congruent)	All numbers different (incongruent)
3/4 vs. 2/4	1/6 vs. 1/8	1/3 vs. 5/7	2/3 vs. 4/9
6/7 vs. 2/7	3/7 vs. 3/5	6/8 vs. 2/5	5/8 vs. 3/4
3/9 vs. 5/9	2/4 vs. 2/9	1/4 vs. 2/6	2/9 vs. 1/3

dings. Each video was followed by two questions used to probe children's understanding of specific contents.

At the end of the session, children answered the ratio comparison task. For each item, they were presented with a pair of ratios (one on each side of the screen) and asked to judge which of them is "more convenient" by pressing the keys Q or P if the left or right ratio, respectively, was the most convenient. Children had no time limit for answering. In order to prevent children having problems with reading comprehension, all questions were presented both in text and in audio. Children were unable to respond before the audio finished.

Data analysis

Accuracy data for the probe questions and the ratio comparison test were analyzed using the R version 3.1.2 (http://www.r-project.org/). Response times were deemed meaningless and not considered in the analysis, because children were not allowed to answer test items before the corresponding audio finished.

Children were grouped in clusters using the k-means clustering algorithm. This method might result in suboptimal clustering due to its dependence on the initial solution (which can be either randomly chosen or provided by the user). To avoid this problem, the algorithm was run on the same data 1000 times using random initial solutions, and only the best clustering of all was kept.

Results

Understanding of the material

Children answered ten questions to assess their understanding of the audiovisual material. These questions could be divided into three categories: five questions about the mapping between ratios and messages and graphical representations, three questions about equivalence of ratios, and two questions of comparing ratios (one where the ratios shared the same number of days and another with the same number of candies). The average score was 79% in mapping and representation ($t(39) = 7.5$, $p < .0001$), 45% in ratio equivalence ($t(39) = -.9$, $p = .35$), and 75% in ratio comparison ($t(39) = 4.9$, $p < .0001$). Children thus succeeded in learning how to map and represent ratios and how to compare pairs of them when they share a common component, but they failed to learn equivalence of ratios.

Ratio comparison test

Overall, children had a mean score of 65% correct answers (SD = 18%), significantly above the chance level of 50% ($t(39) = 5.4$, $p < .0001$). This provides further evidence that, at least in average, children were able to interpret adequately the novel symbols. Separate scores for each of the four item types (Table 2) were then computed and submitted to a logistic regression with congruency and the presence/absence of a common component as fixed factors and children as a random factor. This analysis revealed a significant effect of congruency (OR = -1.0, $p = .0005$), with congruent items being answered more correctly than incongruent items (average scores of 76% and 54%, respectively); a trend towards significance for the presence/absence of common components (OR = .58, $p =.06$), with items sharing a common component being answered more correctly than items with all numbers different (average scores of 70% and 60%, respectively); and no interaction (OR = -.15, $p = .72$).

Clustering analysis

Individual differences in children's understanding were assessed by means of a clustering analysis, considering their accuracy scores for the four item types. This analysis revealed the existence of three groups of children that explained 67.3% of the total variance. Table 2 shows the scores of each group for each item type. Cluster A, the largest with 17 out of the 40 children, had high scores for all item types but for incongruent pairs with all numbers different. This cluster also had the highest overall score. The second most numerous cluster (Cluster B, N = 14) was composed by children who answered mostly guided by the congruency or incongruency of each item according to the NNB account: Congruent items were mostly correct, whereas incongruent items were mostly incorrect. Finally, the last cluster (Cluster C, N = 19) presented a relatively good overall score but an unexpected pattern, with incongruent ratio pairs being answered close to ceiling levels, congruent items with all numbers different mostly incorrect, and surprisingly, not so good scores in the easiest item type (ratio pairs with the same number of days).

Figure 2 presents histograms of correct responses per item type for the full sample and for each cluster. This shows that Cluster B is characterized by strong congruency effects: Twelve out of the 14 children in this cluster performed above 50% on congruent items and below 50% on incongruent items, and no other children in the sample fulfill these criteria. Similarly, Cluster C is perfectly characterized by performance above 50% on incongruent items and below 50% on congruent items with all numbers different. Cluster A, instead, may be characterized by scores in two specific item types, as 15 out of the 17 children in this cluster had scores above 50% in both the second (same number of candies) and third item (congruent with all numbers different) types. Again in this case, no other children in the sample fulfilled these criteria.

Tabla 2. List of items of the ratio comparison task, classified by item type.

Group	Same number of days (congruent)	Same number of candies (incongruent)	All numbers different (congruent)	All numbers different (incongruent)	Total score
Full sample (N = 40)	81 %	59 %	71 %	49 %	65 %
Cluster A (n = 17)	82 %	80 %	84 %	63 %	77 %
Cluster B (n = 14)	83 %	7 %	95 %	5 %	48 %
Cluster C (n = 9)	74 %	100 %	7 %	93 %	69 %

Discussion

The present research aimed at understanding whether children's intuitions about ratios is able to support performance in a ratio comparison, and to explore if this approach elicits a NNB such as the one many studies have documented for fractions at different ages and levels of expertise (Gómez et al., 2014; Obersteiner et al., 2013; Vamvakoussi et al., 2012; Van Eeckhoudt, 2013). The data shed light on both questions. Children's intuitive reasoning was, in average, a successful scaffold for answering the ratio comparison task. The fact that children learned successfully the meaning of the novel symbols but failed to learn equivalence (e.g. the equivalence of 2:3 and 4:6), suggests that children's responses in the ratio comparison task were mostly driven by intuitive reasoning rather than by the use of compu-

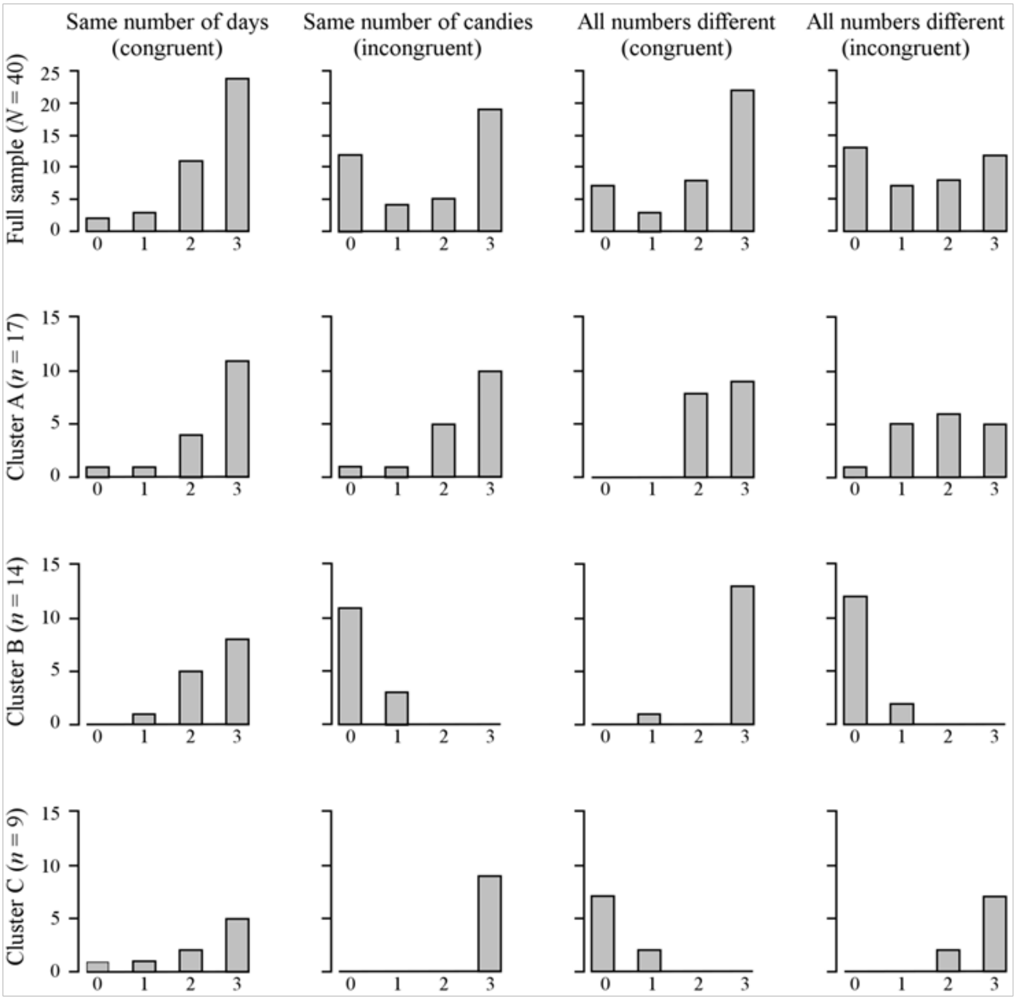

Figura 1. Histograms showing the number of children per group having 0, 1, 2, or 3 correct answers per item type. Panels are organized in columns, corresponding from left to right to: items with the same number of days, items with the same number of candies, congruent items with all numbers different, and incongruent items with all numbers different. The top row presents histograms for the full sample, and the other rows present those of each of the three clusters. For instance, the top-left histogram shows that for items with the same number of days, the distribution of children's correct answers in the full sample was: 2 children answered 0 items correctly, 3 children answered 1 item correctly, 11 children answered 2 items correctly, and 24 children answered all 3 items correctly.

tationally complex strategies. Furthermore, a clustering analysis revealed that children showed distinct patterns of intuitive reasoning, not all of them being compatible with an adequate concept of ratio.

A majority group (cluster A, 43% of the sample) compared ratios successfully across all item types, indicating that intuitive reasoning via ratios may be successful as a pedagogical tool for introducing ratio and proportion. Still, 35% of children (cluster B) reasoned mostly based on the natural numbers composing the ratios (a strong form of NNB, cf. Gómez et al., 2014), disregarding the relations between numbers of candies and days presented in each message and focusing only on comparing the numbers across the two presented messages. These children, if asked to justify their decisions, would probably give explanations where ratios simply consist in two independent numbers. Stafylidou and Vosniadou (2004) presented and considered this explanatory framework as the most basic one, showing that it is still used by 30% of children in 5th grade. Finally, the behavior of children in the third group (22% of the sample) also departed from the predictions of the NNB. If anything, they showed a reversed bias: scores for incongruent items were higher than those for congruent items (in opposition to, e.g., Gómez et al., 2014; Van Eeekhoudt, 2013). This pattern of answers suggests that they might have focused exclusively on the number of days presented in each message, choosing whenever possible the message with the smallest number of days associated. A possible account for this group's reasoning might go beyond ratio comparison per se, as they seemingly deemed more convenient to choose that candies are delivered sooner even if that means receiving fewer candies. Such possibility suggests that these children interpreted the ratio comparison task as an economical decision rather than anything ratio-based (indeed, children facing such decisions seem to prefer shorter delays than higher rewards, e.g. Green, Fry, & Myerson, 1994). It is not uncommon that children use different interpretations or intuitions than those expected by educators and researchers (e.g., Van Den Brink & Streefland, 1979), which leads to one limitation of the present study: the absence of individual interviews or similar methods of inquiry allowing confirmation of how children were actually reasoning. Still, the cluster-based analysis provides an important first step in that direction by grouping children according to their patterns of answers. Further research is also needed to discover whether children's clusters of membership predict how they will reason about fractions later, and to explore continuities and discontinuities in the transition from intuitive ratio concepts to formal ones.

These results also have implications for the teaching of ratios and fractions. First, biased children (those in Cluster B) might benefit from highlighting the relevance of integrating both natural numbers within each ratio into a holistic element and explicit discouraging of simple comparison of the natural numbers across ratios. Hence, activities such as mapping ratios and fractions onto a number line might prove useful for them. Second, the data showed that the same teaching material can be interpreted in a diversity of ways by children. Such differences underline the need of presenting and explaining ratios and fractions using a variety of ways and/or metaphors, so as to minimize the chance of children drawing wrong interpretations or generalizations.

Referencias

Aksu, M. (1997). Performance in dealing with fractions. *The Journal of Educational Research*, *90*(6), 375-380.

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*(4), 247-253.

Depaepe, F., Torbeyns, J., Vermeersch, N., Janssens, D., Janssen, R., Kelchtermans, G., Verschaffel, L., & Van Dooren, W. (2015). Teachers' content and pedagogical content knowledge on rational numbers: A comparison of prospective elementary and lower secondary school teachers. *Teaching and Teacher Education*, *47*, 82-92.

Gómez, D. M., Jiménez, A., Bobadilla, R., Reyes, C., & Dartnell, P. (2014). Exploring fraction comparison in school children. In Oesterle, S., Liljedahl, P., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36, Vol. 3, pp. 185-192. Vancouver, Canada: PME.

Green, L., Fry, A. F., & Myerson, J. (1994). Discounting of delayed rewards: A life-span comparison. *Psychological Science*, 5(1), 33-36.

Izsák, A., Orrill, C. H., Cohen, A. S., & Brown, R. E. (2010). Measuring middle grades teachers' understanding of rational numbers with the Mixture Rasch Model. *The Elementary School Journal*, *110*(3), 279-300.

Jiménez, A. (2012). *Modelación de procesos cognitivos con aplicaciones en educación matemática*. Tesis para optar al título de Ingeniero Civil Matemático, Universidad de Chile, Santiago, Chile. Available online in http://www.tesis.uchile.cl/handle/2250/112312 (last access February 12, 2015).

Lamon, S. J. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education*, *24*(1), 41-61.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27-52.

Obando, G., Vasco, C. E., & Arboleda, L. C. (2014). Enseñanza y aprendizaje de la razón, la proporción y la proporcionalidad: Un estado del arte. *Revista Latinoamericana de Investigación en Matemática Educativa*, *17*(1), 59-81.

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, *28*, 64-72.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691-697.

Singer, Kohn, & Resnick (1997). Knowing about proportions in different contexts. In Nunes & Bryant (Eds.), Learning and teaching mathematics: An international perspective. Hove, England: Psychology Press.

Stafylidou, S., Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*, 503-518.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior*, *31*, 344-355.

Van Den Brink, J., & Streefland, L. (1979). Young children (6-8): Ratio and proportion. *Educational Studies in Mathematics*, *10*(4), 403-420.

Van Dooren, W., Lehtinen, E., & Verschaffel, L. (in press). Unraveling the gap between natural and rational numbers. *Learning and Instruction*, doi: 10.1016/j.learninstruc.2015.01.001

Van Eeckhoudt, K. (2013). *1/7>1/6? De natural number bias bij het vergelijken van breuken. Een reactietijdstudie bij kinderen uit het zesde leerjaar*. Unpublished master's thesis, University of Leuven, Leuven, Belgium.

Van Steenbrugge, H., Lesage, E., Valcke, M., & Desoete, A. (2014).

Preservice elementary school teachers' knowledge of fractions: A mirror of students' knowledge? *Journal of Curriculum Studies*, *46*(1), 138-161.

David Maximiliano Gómez. PhD in Mathematical Modeling and PhD in Neuroscience, Research Associate and Manager of the Laboratory of Neuroscience and Cognition of the Center for Advanced Research in Education of the University of Chile. His research focuses on the cognitive and neural foundations of mathematical concepts and their learning. Some topics of interest: rational numbers, acquisition of math expertise.

Pablo Dartnell. PhD in Mathematics, Associate Professor of the Department of Mathematical Engineering, the Center for Mathematical Modeling, and the Center for Advanced Research in Education of the University of Chile. His research focuses on the teaching and learning of mathematics at all educational levels. Some topics of interest: cognitive variables that affect mathematics learning, use of metaphors in school mathematics.

Acknowledgement. The authors are grateful to the students, parents, teachers, and school personnel who made this work possible. This research was funded by CONICYT Basal Funding for Centers of Excellence, grant FB0003.

Correspondence. David Maximiliano Gómez. Centro de Investigación Avanzada en Educación (CIAE). Universidad de Chile. Periodista José Carrasco Tapia 75; 8330014 Santiago (Chile); E-mail: dgomez@ciae.uchile.cl

Fecha de recepción: 21/7/2015 Fecha de revisión: 7/9/2015 Fecha de aceptación: 13/10/2015